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Combined Linear Quadratic Gaussian and H_∞ Control of a Benchmark Problem

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A combined linear quadratic Gaussian and H_∞ design method is applied to a benchmark problem. Robust controllers are derived that minimize an upper bound of a quadratic performance index subject to an H_∞ norm bound on a disturbance transfer function matrix. Real parameter variations are included in the design through the addition of fictitious weighted disturbances. Three design cases, each with different robustness, performance, and disturbance rejection requirements, are considered for the benchmark problem. Uncertain parameters and noncollocation of the sensor and actuator make the problem nontrivial. Compensators are found that meet the requirements with reasonable control effort, controller complexity, and noise rejection.

Introduction

A LARGE number of papers have recently been written on the application of different robust control design methods to the benchmark problem proposed by Wie and Bernstein.[1] Because of the noncollocation of the sensor and actuator, most methods for deriving controllers with reasonable complexity and control effort yield a nonminimum phase compensator for this problem.[2–6] The requirement of reasonable control effort is a major constraint on achievable performance and robustness. Most designs that significantly beat the prescribed goals for settling time and disturbance rejection use excessive control effort.[7–9]

The objective of this paper is to design robust controllers for a benchmark problem using a combined linear quadratic Gaussian (LQG) and H_∞ synthesis technique. The approach is to find a strictly proper, stable controller that minimizes an upper bound of the H_2 norm of a transfer function matrix subject to a constraint on the H_∞ norm of a different transfer function matrix for a system with real parameter variations. This design technique allows multiple design goals such as performance, robustness, and disturbance rejection to be addressed simultaneously.

Combined Linear Quadratic Gaussian and H_∞ Synthesis

Consider an nth-order linear time-invariant system of the form

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + \Gamma_1 w_1(t) \quad (1)$$

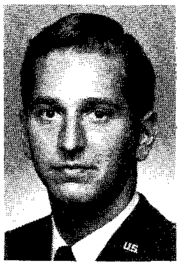
$$y(t) = (C + \Delta C)x(t) + \Gamma_2 w_2(t) \quad (2)$$

$$z_\infty = (E_\infty + \Delta E_\infty)x(t) \quad (3)$$

where u is an m-dimensional control vector; w_1 and w_2 are p_1- and p_2-dimensional disturbance vectors, respectively; z_∞ is a q_∞-dimensional performance variable; and y is an r-dimensional sensor measurement. ΔA, ΔB, and ΔC represent real parameter uncertainty in the system matrices. An nth-order compensator $K(s)$ is represented by the state equations

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \quad (4)$$

$$u(t) = C_c x_c(t) \quad (5)$$



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Now the closed-loop system can be given by

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}\tilde{w}(t) \tag{6}$$

where

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \qquad \Delta\tilde{A} = \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & 0 \end{bmatrix} \tag{7}$$

$$\tilde{D} = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & B_c\Gamma_2 \end{bmatrix} \tag{8}$$

$$\tilde{x}(t) = [x^T(t)\ x_c^T(t)]^T, \qquad \tilde{w}(t) = [w_1^T(t)\ w_2^T(t)]^T \tag{9}$$

The constrained LQG optimization problem is to find a strictly proper, constant controller that produces the following closed-loop properties for all ΔA, ΔB, and ΔC:

1) The closed-loop system is asymptotically stable.
2) The disturbance transfer function matrix given by

$$H(s,\Delta\tilde{A}) = (E_\infty + \Delta E_\infty)\,(sI - \tilde{A} - \Delta\tilde{A})^{-1}\tilde{D} \tag{10}$$

is norm bounded by the relation

$$\|H(s,\Delta\tilde{A})\|_\infty \le \gamma \tag{11}$$

for a positive constant γ.

3) An upper bound on a H_2 performance criterion

$$J(\Delta\tilde{A}) = \lim_{t\to\infty} E[x^T(t)R_1x(t) + u^T(t)R_2u(t)] \tag{12}$$

is minimized.

A solution for this problem has been found for the case where ΔA and ΔC exist, while $\Delta B = 0$.[10] The solution for $\Delta B \neq 0$ and $\Delta C = 0$ has been found in a dual-design approach.[11] The real parameter uncertainty in the A and C matrices is represented by

$$\Delta A = \sum_{i=1}^{p} D_iM_iN_iE_i, \qquad \Delta C = \sum_{i=1}^{p} F_iM_iN_iE_i \tag{13}$$

where D_i, F_i, E_i, M_i, and N_i are matrices of appropriate dimension; D_i, F_i, and E_i define the structure of the uncertainty; and M_i and N_i are uncertain matrices subject to the constraints

$$M_iM_i^T \le \overline{M}_i, \qquad N_iN_i^T \le \overline{N}_i, \qquad \overline{M}_i\overline{N}_i = 1 \tag{14}$$

The real parameter uncertainties are included by overbounding their variations and modeling their effects as disturbances acting on a surrogate system. The parameters that make up this system are

$$R'_{1\infty} = E_{1\infty}^TE_{1\infty} + 2\mu_c\sigma_{\max}(C)I + \gamma^2\sum_{i=1}^{p} E_i^T\overline{N}_iE_i$$

$$\mu_c = \sigma_{\max}(\Delta C) \tag{15}$$

$$V' = \begin{bmatrix} V_1' & V_{12}'B_c^T \\ B_cV_{21}' & B_cV_2'B_c^T \end{bmatrix} \tag{16}$$

where

$$V_1' = \Gamma_1\Gamma_1^T + \sum_{i=1}^{p} D_i\overline{M}_iD_i^T, \qquad V_{12}' = \sum_{i=1}^{p} D_i\overline{M}_iF_i^T$$

$$V_{21}' = \sum_{i=1}^{p} F_i\overline{M}_iD_i^T, \qquad V_2' = \Gamma_2\Gamma_2^T + \sum_{i=1}^{p} F_i\overline{M}_iF_i^T \tag{17}$$

In most cases for each i, either D_i or F_i is zero. Therefore, V' is normally diagonal and the surrogate system takes the form in Fig. 1. It has been shown[10] that a controller that achieves the desired properties for the surrogate system also achieves these properties for the actual system. The robust synthesis problem is shown in Ref. 10 to be satisfied by the controller

$$A_c = A + BC_c - B_cC + \gamma^{-2}Q'R'_{1\infty} \tag{18}$$

$$B_c = (Q'C^T + V'_{12})(V'_2)^{-1} \tag{19}$$

$$C_c = -R_2^{-1}B^TP \tag{20}$$

where Q', $\hat{Q}$, and P are solutions of

$$\begin{aligned} 0 = {} & [A - V'_{12}(V'_2)^{-1}C]Q' + Q'[A - V'_{12}(V'_2)^{-1}C]^T \\ & + Q'[\gamma^{-2}R'_{1\infty} - C^T(V'_2)^{-1}C]Q' \\ & + V'_1 - V'_{12}(V'_2)^{-1}V'_{21} \end{aligned} \tag{21}$$

$$\begin{aligned} 0 = {} & [A - BR_2^{-1}B^TP + \gamma^{-2}Q'R'_{1\infty}]\hat{Q} \\ & + \hat{Q}[A - BR_2^{-1}B^TP + \gamma^{-2}Q'R'_{1\infty}]^T + \gamma^{-2}\hat{Q}R'_{1\infty}\hat{Q} \\ & + (Q'C^T + V'_{12})(V'_2)^{-1}(Q'C^T + V'_{12})^T \end{aligned} \tag{22}$$

$$\begin{aligned} 0 = {} & [A + \gamma^{-2}(Q' + \hat{Q})R'_{1\infty}]^TP \\ & + P[A + \gamma^{-2}(Q' + \hat{Q})R'_{1\infty}] + R_1 - PBR_2^{-1}B^TP \end{aligned} \tag{23}$$

and the quadratic performance index has the bound

$$J(\Delta\tilde{A}) \le \mathrm{tr}[(Q' + \hat{Q})R_1 + \hat{Q}PBR_2^{-1}B^TP] \tag{24}$$

This method allows for the combination of LQG and H_∞ design goals to be put within a common framework. Some of the advantages as well as the disadvantages of the two methods are apparent in the combined synthesis approach. The manner in which this new method is applied is highly dependent on the design task. The problem can be formulated such that robustness to both unmodeled dynamics and real parameter variations is achieved.[10] It can also be set up to perform loop shaping in an LQG/LTR type approach.[12] In this paper, a more straightforward approach is taken to the selection of design parameters in order to achieve specific performance and robustness goals. The primary advantages of combined LQG and H_∞ are the freedom to tailor the method to meet the requirements of a particular problem and the ability to achieve robustness to real parameter variations. The primary disadvantages are the need for design iterations and the requirement to solve three coupled Riccati equations. Like many multivariable design approaches, the controller is of the same order as the plant plus that of any weighting filters used in the problem.

Benchmark Problem for Robust Control

An undamped two-mass/spring system is shown in Fig. 2. The sensor and actuator are noncollocated in this generic

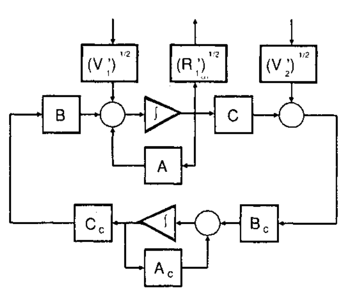

Fig. 1 Surrogate system.

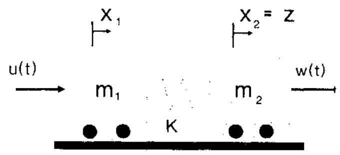

Fig. 2 Benchmark spring-mass system.

model of an uncertain dynamical system. A representative state-space model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w$$

$$y = x_2 + v \qquad z = x_2 \tag{25}$$

The control force u acts on body 1, and the sensor output y is the position of body 2; z is the performance variable, w the plant disturbance, and v the sensor noise. The plant parameters k, m_1, and m_2 all have the nominal value of 1.

Three problems are posed for the design of constant gain, linear, output feedback controllers of the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{26}$$

$$u(t) = C_c x(t) + D_c y(t) \tag{27}$$

The control system should tolerate reasonable noise measurement signals $v(t)$, have reasonable performance/stability robustness with reasonable bandwidth, exhibit acceptable control effort levels, and have reasonable controller complexity.

Robust Controller Design Results

Problem 1

Stability of the closed-loop system must be achieved for $m_1 = m_2 = 1$, and $0.5 < k < 2.0$. For a unit impulse disturbance $w(t)$ applied to the nominal system, the performance variable z must have settling time of about 15 s. For the sake of this design, we define settling time as the time to reach and stay within zero +/− 10% of the maximum displacement.

The benchmark problem for robust control design falls easily within the framework of combined LQG and H_{∞} synthesis. The design model is chosen with the parameters $m_1 = m_2 = 1$, and $k = 1.25$. The real parameter uncertainty in the A matrix is represented in the form of Eqs. (13) as

$$\Delta A = D_1 M_1 N_1 E_1 \tag{28}$$

$$D_1 = [0 \;\; 0 \;\; 0.75 \;\; -0.75]^T, \qquad E_1 = [-1 \;\; 1 \;\; 0 \;\; 0] \tag{29}$$

$$\overline{M}_1 = 110, \qquad \overline{N}_1 = 1/\overline{M}_1 \tag{30}$$

If a solution can be found to the three coupled Riccati equations that define the controller, the closed-loop system will be guaranteed stable for spring constants from 0.5 to 2.0. Reduced scaling of D_1 may also yield systems with the prescribed robustness because the uncertainty is captured by a conservative bound in the design formulation.

The disturbance transfer matrix is defined by the parameters

$$E_{\infty} = C, \qquad \Gamma_1 = [0 \;\; 0 \;\; 0 \;\; 1]^T, \qquad \Gamma_2 = 1 \tag{31}$$

and the H_2 performance index weights are chosen as

$$R_1 = C^T C, \qquad R_2 = 0.005 \tag{32}$$

By adjusting the H_2 control weight and the noise weightings in the surrogate system, a compromise is reached between parameter robustness, closed-loop performance, and control effort. The H_2 weights are chosen in a similar fashion to LQG design, with R_1 as the quadratic weighting of state variables and R_2 as the control weighting. The disturbance input distribution matrix Γ_1 is varied to achieve the desired level of rejection to plant disturbances. Γ_2 is increased to achieve robustness to unmodeled dynamics by providing a weight on the complementary sensitivity function.[10] Achieving the final controller is an iterative process due to the inherent tradeoffs that are present when attempting to satisfy multiple design goals. The result is the following stable, fourth-order, strictly proper, nonminimum phase compensator

$$K_1(s) = \frac{u(s)}{y(s)} = \frac{-6.87s^3 + 93.14s^2 - 27.34s - 4.08}{s^4 + 7.49s^3 + 30.55s^2 + 73.88s + 93.80} \tag{33}$$

with poles at $s = -1.0274 +/- 3.0025j$ and $s = -2.7179 +/- 1.3882j$. There are nonminimum phase zeros at $s = 0.4123$, 13.2590, and a stable zero near the origin at $s = -0.1086$.

The resulting closed-loop system is stable for

$$0.381 \leq k \leq 3.195 \tag{34}$$

Figure 3 shows a root migration plot for the closed loop system with spring constants from 0.5 to 2.0. Notice that the poles corresponding to the rigid-body mode change very little as a function of the parameter k. The nominal system has a gain margin of 2.67 dB at 0.52 rad/s and a phase margin of 21.8 deg at 0.27 rad/s. Sensor noise should not be a problem as the closed-loop system exhibits a high rolloff rate above 3 rad/s where any such noise would likely exist. Figure 4 shows that noise signals are highly attenuated in the upper range of frequencies.

The unit impulse responses of z for spring constants $k = 0.5$, 1.0, and 2.0 are shown in Fig. 5 with corresponding control histories in Fig. 6. A settling time of about 15 s has been achieved for the nominal system with reasonable controller effort, $\|u[t)\|_1 < 1$ for $k = 1$.

Problem 2

Stability robustness is maximized with respect to the parameters m_1, m_2, and k. The performance variable z must

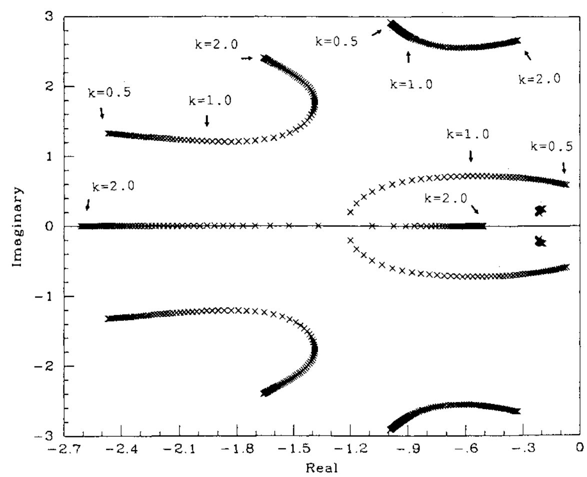

Fig. 3 Design 1 closed-loop poles for changing k.

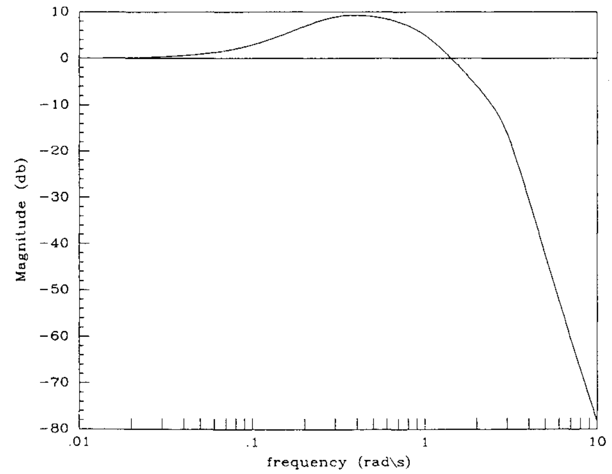

Fig. 4 Complementary sensitivity function for design 1.

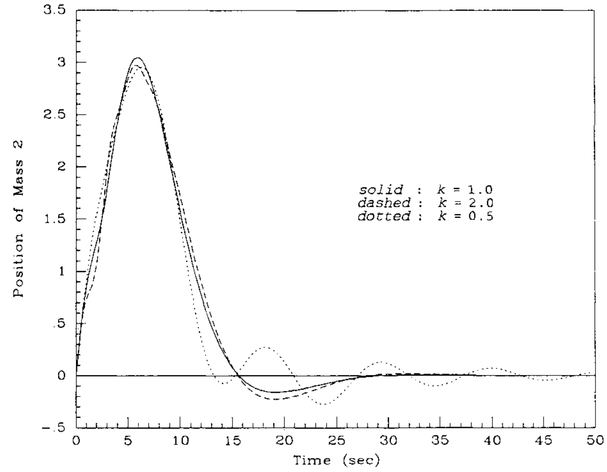

Fig. 5 $x2$ response to unit impulse $w(t)$ for design 1.

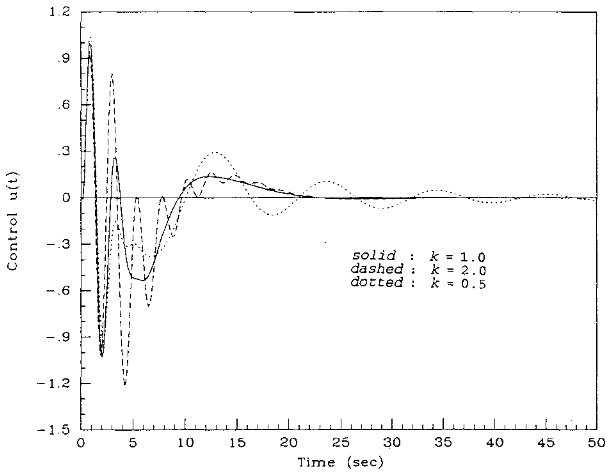

Fig. 6 $u(t)$ response to unit impulse $w(t)$ for design 1.

have a settling time of about 15 s in response to a unit impulse disturbance applied to the nominal system.

The robust synthesis problem now includes three uncertain parameters, k, m_1, and m_2, which appear in the state matrices in nonlinear combinations. The goal is to design $K(s)$ to maximize the size of the cube in three-dimensional parameter space where k, m_1, and m_2 can vary independently and simultaneously while closed-loop stability is maintained. The combined LQG and H_∞ formulation is similar to design 1 because the uncertainty enters the A matrix at the same locations. The input and disturbance distribution matrices are now functions of m_1 and m_2, and so real parameter uncertainty ΔB and $\Delta\Gamma_1$ must be considered. A new design parameter is defined to put these uncertainties into a form that can be captured in a combined LQG and H_∞ solution:

$$\eta = k = \frac{1}{m_1} = \frac{1}{m_2} \tag{35}$$

This reformulation captures one limiting case of parameter variations, masses and spring constant moving in opposite directions. Although this may not be the worst case for a particular controller solution, a broad class of parameter uncertainties are captured for design purposes.

The state matrices may now be rewritten for design formulation:

$$A_\eta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\eta^2 & \eta^2 & 0 & 0 \\ \eta^2 & -\eta^2 & 0 & 0 \end{bmatrix}, \quad B_\eta = \begin{bmatrix} 0 \\ 0 \\ \eta \\ 0 \end{bmatrix}, \quad \Gamma_{1\eta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \eta \end{bmatrix}$$

$$C_\eta = [0 \quad 1 \quad 0 \quad 0] \tag{36}$$

Since our design algorithm cannot directly account for $\Delta\Gamma_1$, we derive an equivalent state-space form that moves the uncertain parameter η from the B and Γ_1 matrices to the C matrix:

$$A'_\eta = A_\eta, \qquad B'_\eta = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \Gamma'_{1\eta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C'_\eta = [0 \quad \eta \quad 0 \quad 0] \tag{37}$$

Now the problem is one for which all of the real parameter uncertainty can be handled in the combined LQG and H_∞ framework. The structure of the uncertainty is defined by

$$\Delta A = D_1M_1N_1E_1 + D_2M_2N_2E_2$$

$$\Delta C = F_1M_1N_1E_1 + F_2M_2N_2E_2 \tag{38}$$

$$D_1 = [0 \quad 0 \quad 1 \quad -1]^T, \qquad D_2 = [0 \quad 0 \quad 0 \quad 0]^T \tag{39}$$

$$F_1 = 0, \qquad F_2 = 1 \tag{40}$$

$$E_1 = [-1 \quad 1 \quad 0 \quad 0] \times \varepsilon_1, \qquad E_2 = [0 \quad 1 \quad 0 \quad 0] \times \varepsilon_2 \tag{41}$$

$$\overline{M}_i = 500, \qquad \overline{N}_i = 1/\overline{M}_i, \qquad i = 1, 2 \tag{42}$$

The design parameters E_∞, Γ_2, R_1, and R_2 are assigned the same values as in design 1. The new parameters ε_1 and ε_2 are increased until the maximum parameter robustness is achieved. The result is the fourth-order nonminimum phase compensator

$$K_2(s) = \frac{u(s)}{y(s)}$$

$$= \frac{-23.31s^3 + 105.29s^2 - 29.43s - 3.86}{s^4 + 7.71s^3 + 32.23s^2 + 80.27s + 110.36} \tag{43}$$

with poles at $s = -0.9722 +/- 3.0603j$ and $s = -2.8831 +/- 1.5466j$. Two nonminimum zeros exist at $s = 0.4063$ and 4.2079, and one minimum phase zero lies near the origin at $s = -0.0968$. This controller is very similar to the design 1 compensator due to the similarity of design goals and uncertainty structure. The primary difference is that an increase in controller effort is required to achieve robustness for a broader range of real parameter uncertainties.

The controller creates a stable region in three-dimensional parameter space defined by the cube

$$0.676 \leq k, m_1, m_2 \leq 1.47 \tag{44}$$

The upper limit is defined by the case where $\eta = 1.47$, and the lower limit is set by $k = m_1 = m_2 = 0.676$. Notice that, when the parameters change as $k = m_1 = m_2$, the A matrix remains constant and the B matrix varies as $1/m_1$. The lower limit, therefore, also defines the inverse of the gain margin (GM): GM = 1/0.676 = 3.40 dB at 0.51 rad/s. The phase margin for this design is 24.8 deg at 0.23 rad/s. As in design 1, sensor noise should not be a problem because of a high gain rolloff rate for the closed-loop system at the frequencies of concern.

If the masses are held at their nominal values, the system is stable for spring constants of

$$0.311 \leq k \leq 2.56 \tag{45}$$

If the spring constant and one of the mass values are held at 1, the controller stabilizes the system for

$$0.303 \leq m_{1 \text{ or } 2} \leq \infty \tag{46}$$

Figure 7 shows the response of the closed-loop plant to a unit impulse for three values of η. The nominal system has a settling time of about 15 s. Figure 8 shows the corresponding control histories. The maximum control effort for $k = m_1 = m_2$ has increased from the design 1 impulse response. This increase illustrates a tradeoff between the design goals of reasonable control effort and real parameter robustness.

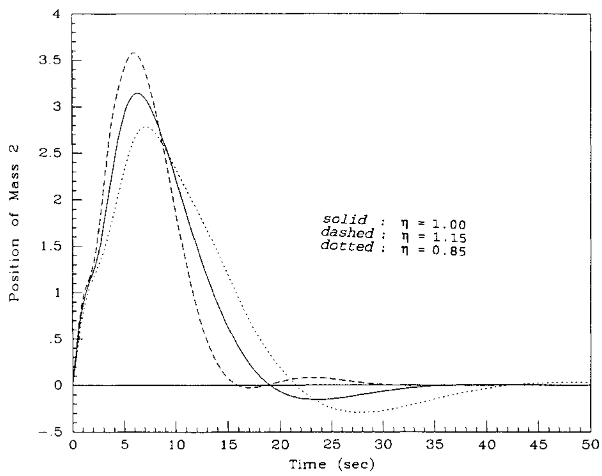

Fig. 7 $x2$ response to unit impulse $w(t)$ for design 2.

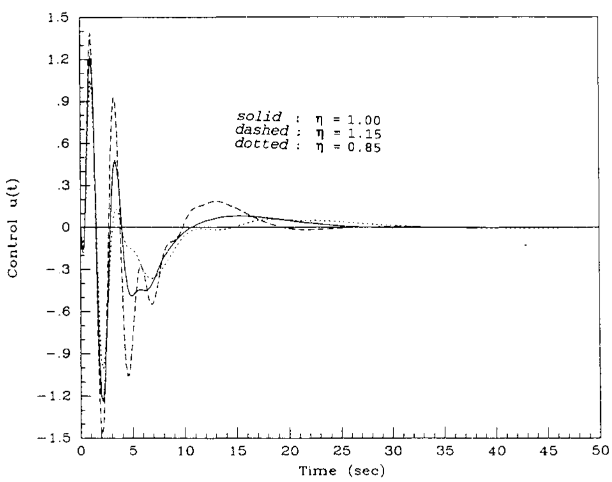

Fig. 8 $u(t)$ response for unit impulse $w(t)$ for design 2.

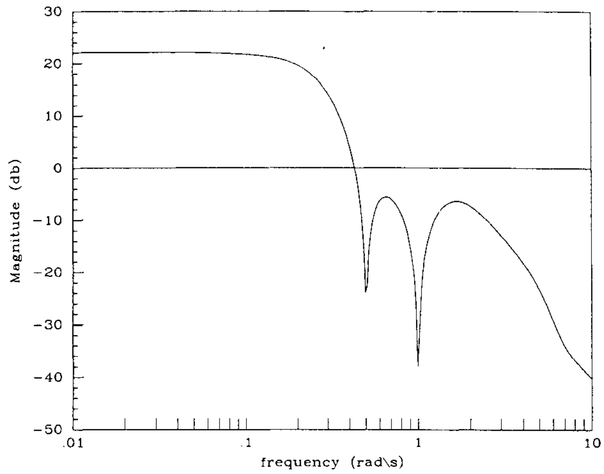

Fig. 9 Disturbance response function for design 3.

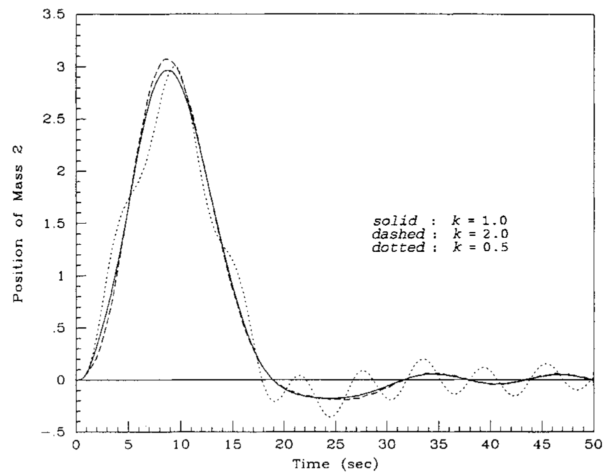

Fig. 10 $x2$ response to $w(t) = \sin(0.5{*}t)$ for design 3.

Problem 3

A sinusoidal disturbance $w(t)$ of frequency 0.5 rad/s and unknown amplitude and phase is applied to the system. Asymptotic rejection of the disturbance should be achieved at the performance variable z, with a 20-s settling time for $m_1 = m_2 = 1$, $0.5 < k < 2.0$.

The task of asymptotic disturbance rejection is solved by the reformulation of the combined LQG and H_∞ synthesis to include a special weighting filter. The purpose of this weighting filter is to force the controller to notch out the disturbance while maintaining stability, robustness, and performance. This inverse notch weighting or spike filter has the form

$$\dot{x}_w(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta_2\omega_n \end{bmatrix} x_w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(t)$$

$$y_w(t) = [0 \quad 2\omega_n(\zeta_1 - \zeta_2)]x_w(t) + y(t) \tag{47}$$

where

$$\omega_n = 0.5 \text{ rad/s}, \qquad \zeta_1 = 0.707, \qquad \zeta_2 = 0.01 \tag{48}$$

ζ_2 was chosen as 0.01 instead of 0.00 in order to achieve better convergence in the design algorithm.

The filter state equations are combined with the plant state equations to form the controller synthesis model:

$$\dot{x}_s(t) = \begin{bmatrix} A & 0 \\ B_w C & A_w \end{bmatrix} x_s(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \Gamma_1 \\ 0 \end{bmatrix} w_1(t)$$

$$y_s(t) = [C \quad 0]x_s(t) + v$$

$$z_s(t) = [C \quad C_w]x_s(t) \tag{49}$$

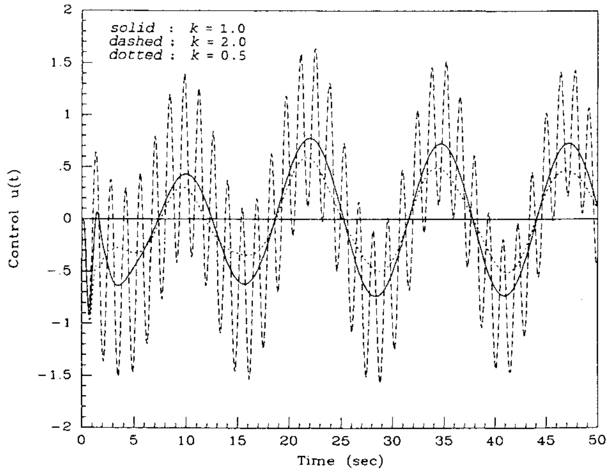

Fig. 11 $u(t)$ response to $w(t) = \sin(0.5*t)$ for design 3.

The uncertainty structure for this problem is identical to design 1. The disturbance transfer matrix now includes the filter model due to the redefinition of the performance variable z_∞. The redefined parameters are

$$E_{\infty_3} = [C \quad C_w], \qquad \Gamma_{1_3} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0]^T$$

$$\Gamma_{2_3} = 1 \tag{50}$$

and the H_2 performance weights are now

$$R_1 = E^T_{\infty_3} E_{\infty_3}, \qquad R_2 = 0.0005 \tag{51}$$

The resulting compensator is sixth order:

$$K_3(s) = \frac{-4280s^5 - 1237s^4 - 4898s^3 - 769.6s^2 - 500.2s - 57.26}{s^6 + 17.68s^5 + 159.1s^4 + 873.1s^3 + 2967s^2 + 247.7s + 731.9} \tag{52}$$

The frequency-dependent weighting has forced the compensator to have a very lightly damped pair of poles at $s = -0.0052 +/- 0.5008j$. The remaining compensator poles are at $s = -1.7195 +/- 6.5142j$ and $-7.1155 +/- 3.6948j$. The controller zeros are all minimum phase: $s = -0.0698 +/- 1.0024j$, $-0.0146 +/- 0.3320j$, and -0.1200. Figure 9 is a magnitude plot for the transfer function between the output $y(t)$ and the disturbance input $w(t)$. The notch effect that has been created at 0.5 rad/s produces the desired rejection of sinusoidal disturbances at that frequency. Figure 10 shows the response of the closed-loop system to a sinusoidal disturbance input for three values of spring constant, $k = 0.5$, 1.0, and 2.0. The $k = 1.0$ and 2.0 cases have settling times of about 20 s and show disturbance rejection in excess of 20 dB, as predicted in Fig. 9. The $k = 0.5$ case also shows good disturbance rejection, but a lightly damped flexible mode continues to oscillate past 50 s. The corresponding control histories are presented in Fig. 11. The $k = 2.0$ case shows that a lightly damped, high-frequency controller mode has been excited at this off-design condition. This mode is not observed in the sensor output.

The closed-loop system is stable for

$$0.476 \le k \le 2.007 \tag{53}$$

The nominal system has excellent gain and phase margins (PM): GM = 7.1 dB at 4.5 rad/s and PM = 43.5 deg at 2.2 rad/s. This design exhibits good noise rejection for frequencies above 10 rad/s.

Conclusions

Three stable, strictly proper compensators have been designed for a spring-mass benchmark problem for robust control design. A combined LQG and H_∞ synthesis method was used to minimize an upper bound of a quadratic performance index subject to an H_∞ norm constraint on a disturbance transfer matrix. For the three design tasks considered, satisfactory performance, disturbance rejection, and robustness to real parameter variations has been achieved with reasonable controller effort and complexity.

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